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THE MULTI-DEE CYCLOTRON

by

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INTRODUCTION

The conventional fixed frequency cyclotron is constructed with two accelerating electrodes or dees. Each dee has an angular extent of 180° . The peak voltages from the dees to ground are the same for each dee, but differ in phase by 180° . One recent paper has proposed a cyclotron of two similar dees of angular widths less than 180° , symmetrically spaced, and excited by equal voltages either in or out of phase.¹ Another paper has described a machine with three similar dees of 120° or less, all symmetrically spaced, and with three different voltage phasing modes.² The theory is extended in this paper, giving the energy gain per turn equation for a cyclotron with n -dees. Initially the dees are considered to be similar and symmetrically spaced with equal dee voltages and equal voltage phase shifts. Then the equation is generalized for non-similar dees, non-symmetrically spaced, with non-equal voltages and voltage phase shifts. The restrictions on these several variables are found in order that stable acceleration requirements be met. For all these calculations, it has been assumed that the voltage change at each dee edge is a step function.

THE SYMMETRICAL n -DEE CYCLOTRON

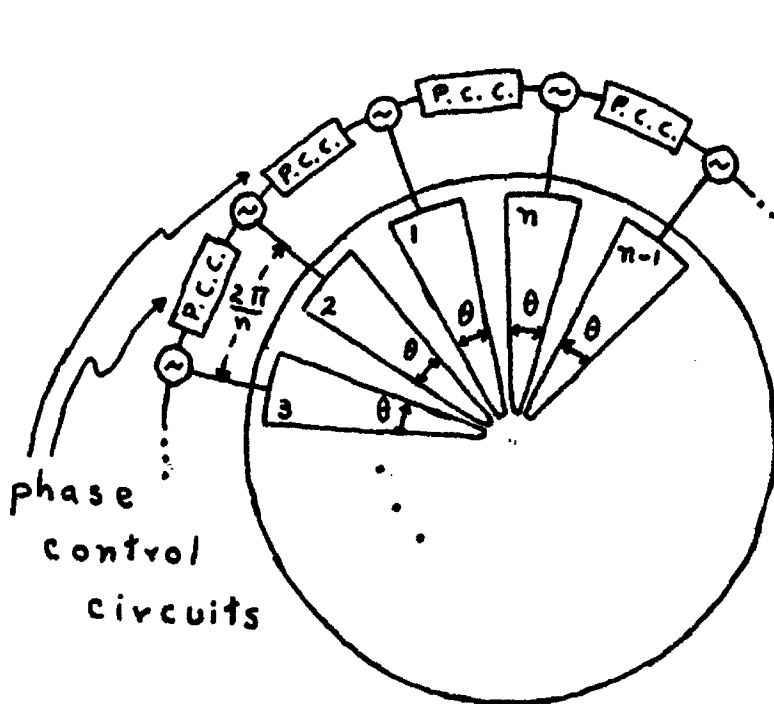
The cyclotron schematically drawn in Figure 1 has n similar dees labeled 1, 2, 3, . . . n . Each has angular width $\phi \leq \frac{2\pi}{n}$ and

¹Jakobson and Schmidt, Phys. Rev., Vol. 93, 303 (1954).

²Jakobson, Heusinkveld, and Ruby, Phys. Rev., Vol. 104, 362 (1956).

centerline separation from the preceding dee of $\frac{2\pi}{n}$. The peak ground to dee voltage are equal and denoted by V . The voltage on each dee lags the preceding dee voltage by a phase shift $\alpha = \frac{2\pi}{n}\pi$ where $K = 0, 1, \dots, (n-1)$. This allows n different modes of acceleration.

Let ω be the angular frequency of the power supply. Then at a time t , the voltage on each dee can be expressed as :



$$\begin{aligned} V_1 &= V_0 e^{i\omega t} \\ V_2 &= V_0 e^{i(\omega t - \alpha)} \\ V_3 &= V_0 e^{i(\omega t - 2\alpha)} \quad (1) \\ &\dots \\ V_n &= V_0 e^{i[\omega t - (n-1)\alpha]} \end{aligned}$$

By denoting the time that the ion crosses the center of the first dee as t_0 , the time that it crosses each successive dee edge is:

$$\begin{aligned} t_1 &= t_0 + \frac{\theta}{\Omega}, \quad t_2 = \\ &t_0 + \frac{2\pi}{n\Omega} - \frac{\theta}{\Omega}, \quad \dots, \\ t_{2n} &= t_0 + \frac{2\pi}{\Omega} - \frac{\theta}{\Omega}. \end{aligned}$$

Figure 1.

$\Omega = \frac{qB}{m}$ is the angular frequency of the ion in its path, where B is the magnetic induction between the pole tips, q is the charge on the ion, and m is the mass of the ion.

The energy gain at a given dee edge is the product of the charge on the ion and the voltage on the dee at the time the ion crosses the dee edge. The energy gain per turn is the scalar sum of the energy gains at each dee edge as the ion makes one complete revolution. The

angle $\phi = \omega t_0$ is the phase difference between ω and Ω as the ion is at the center of the first dee. The imaginary part of the voltage expressions are taken so that the voltage on the dee is zero when the ion crosses the center of the dee for $\phi = 0$.

Carrying out the sum as indicated above, the energy gain per turn becomes:

$$\Delta W = 2qV \sin \frac{\omega}{2\Omega} \operatorname{Re} \left\{ e^{i\phi} \sum_{r=0}^{n-1} e^{irf} \right\} \quad (2)$$

where $f = \frac{2\pi}{n} (\frac{\omega}{\Omega} - K)$. In order for the average energy gain per turn to be different from zero, it can be shown that $f = 2\pi P$ for P an integer. Thus the summation term in equation (2) becomes:

$$\sum_{r=0}^{n-1} e^{irf} = 1 + e^{if} + e^{i2f} + \dots + e^{i(n-1)f} = n. \quad (3)$$

So, in general

$$\Delta W = 2nqV \sin \frac{\omega}{2\Omega} \cos \phi. \quad (4)$$

Furthermore $f = \frac{2\pi}{n} (\frac{\omega}{\Omega} - K) = 2\pi P$ yields the expression

$$\frac{\omega}{\Omega} = nP + K. \quad (5)$$

For a given mode of acceleration, this relation determines the ions which can be accelerated. However, some of these ions which satisfy equation (5) will not be accelerated if the $\sin \frac{\omega}{2\Omega}$ term in equation (4) is zero for that particular value of $\frac{\omega}{\Omega}$.

As is stated in the description of the symmetrical three-dee cyclotron³, so also in the n-dee case, the energy gain drops to zero if the rf phase varies to $\pm \frac{\pi}{2}$, corresponding to $\pm \frac{\Omega \pi}{2\omega}$ angular displace-

³Ibid.

ment of the ion from the center of the dee when the instantaneous voltage is zero. The maximum angular extent of each ion bunch in the cyclotron is $\frac{\Omega}{\omega} \pi$ regardless of the number or size of the dees. The number of bunches present per cycle is $\frac{\omega}{\Omega}$.

If the angular widths of the dees are taken as $\frac{2\pi}{n}$, the energy gain per turn in general increases for an increased number of dees. In the limit, as n approaches infinity, ΔW approaches $\frac{2eV \pi \omega}{\Omega}$. For a finite number of dees of angular width $\frac{2\pi}{n}$, ions of $\frac{\omega}{\Omega}$ ratio equal to nP cannot be accelerated since the voltage phasing mode is $K = 0$. (See Table I.) If the dees are cut back to be somewhat less than $\frac{2\pi}{n}$ in angular extent, then all ions will be accelerated except those for which $\sin \frac{\theta \omega}{2\Omega} = 0$. In accordance with equation (4), the energy gain per turn for the accelerated ions is less for some ions and greater for others, depending on the value of θ and $\frac{\omega}{\Omega}$.

The size of the ion beam does not critically depend on the number of dees. However, it probably will be less for $n > 2$ than in the conventional two-dee machine, since a source considerably off-center will be necessary.

THE NON-SYMMETRICAL n-DEE CYCLOTRON

In order to generalize the above equation (4) for n non-similar dees, which are non-symmetrically spaced with variable voltage and voltage phasing for each dee, the following variables are introduced:

θ_j = the angular width of the j th dee.

ψ_j = the angular separation between the j th and $(j \neq 1)$ th dees.

V_j = the peak ground to dee voltage on the j th dee.

α_j = the voltage phasing lag between the j th and $(j \neq 1)$ th dees.

There are three conditions that must be met by the cyclotron dee construction and excitation:

$$(a) \sum \psi_j = 2\pi, \text{ by the physical geometry of the machine.} \quad (6)$$

$$(b) \sum \alpha_j = 2K\pi, \quad K = 0, 1, \dots, (n-1), \text{ in order to keep phase control.} \quad (7)$$

$$(c) \sum \vec{R}_j = 0, \text{ where } \vec{R}_j = \text{resultant impulse on the ion from both edges of the } j\text{th dee. This condition must be met in order that there be no net impulse in any direction as the ion makes a complete revolution. A net impulse which is not zero would cause precession of the ion from its normal path.} \quad (8)$$

In the following discussion the non-symmetrical n-dee cyclotron is considered for the cases of 2, 3, 4, and N dees.

1. Two non-similar dees, evenly spaced.

Consider the schematic two-dee cyclotron in Figure 2.

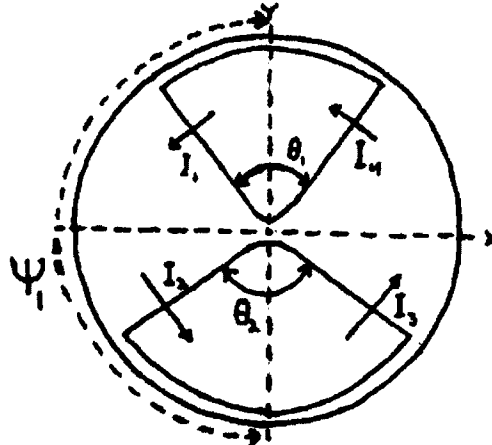


Figure 2.

It will be shown that for maximum energy gain per turn the angle of separation of the dees must be 180° .

Following the same method as used in the symmetrical case, the energy gain per turn is:

$$\Delta W = 2q \left[v_1 \sin \frac{\theta_1 \omega}{2\Omega} \cos \phi + v_2 \sin \frac{\theta_2 \omega}{2\Omega} \cos \left(\phi + \frac{\psi \omega}{\Omega} - \alpha_1 \right) \right] \quad (9)$$

For a machine of given construction, maximum energy gain per turn will be obtained for

$$\frac{\psi_1 \omega}{\Omega} - \alpha_1 = 2s\pi - \phi \quad (10)$$

where $s = 0, \pm 1, \pm 2, \dots$. Since ϕ can be both \neq and $=$ during the acceleration of an ion, near maximum energy gain will be accomplished for both \neq and $=$ values if α_1 is chosen at $\phi = 0$, i.e.,

$$\alpha_1 = \frac{\psi_1 \omega}{\Omega} \neq 2s\pi. \quad (11)$$

Then the maximum energy gain per turn for $\phi = 0$ is:

$$\Delta W_m = 2q \left[V_1 \sin \frac{\theta_1 \omega}{2\Omega} \neq V_2 \sin \frac{\theta_2 \omega}{2\Omega} \right]. \quad (12)$$

The impulse which an ion will receive at each dee edge is found to be

$$|\vec{I}| = \frac{qV_1}{v} \mathcal{I}_m \left\{ e^{i\omega t} \right\} \text{ in magnitude where } v = \text{velocity of the ion.}$$

The direction of the impulse is perpendicular to the dee edge.

In order to satisfy condition (6) above, the summation of the x components and of the y components of the impulses received by the ion during one revolution must each equal zero.

Thus,

$$V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} \cos \phi \neq V_2 \left[\cos \frac{\theta_2}{2} \cos \psi_1 \sin \frac{\theta_2 \omega}{2\Omega} \cos \left(\phi \neq \frac{\psi_1 \omega}{\Omega} - \alpha_1 \right) \right. \\ \left. - \sin \frac{\theta_2}{2} \sin \psi_1 \cos \frac{\theta_2 \omega}{2\Omega} \sin \left(\phi \neq \frac{\psi_1 \omega}{\Omega} - \alpha_1 \right) \right] = 0$$

and

$$V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} \sin \phi - V_2 \left[\sin \frac{\theta_2}{2} \cos \psi_1 \cos \frac{\theta_2 \omega}{2\Omega} \sin \left(\phi \neq \frac{\psi_1 \omega}{\Omega} - \alpha_1 \right) \right. \\ \left. \neq \cos \frac{\theta_2}{2} \sin \psi_1 \sin \frac{\theta_2 \omega}{2\Omega} \cos \left(\phi \neq \frac{\psi_1 \omega}{\Omega} - \alpha_1 \right) \right] = 0. \quad (14)$$

These equations can be rewritten in the more convenient form:

$$V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} \neq V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \cos \left(\frac{\psi_1 \omega}{\Omega} - \alpha_1 \right) \cos \psi_1$$

$$\begin{aligned}
 &= V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \sin \psi_1 \\
 &= \tan \phi \left[V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \cos \psi_1 \right. \\
 &\quad \left. + V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \cos \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \sin \psi_1 \right]
 \end{aligned}$$

and

(15)

$$\begin{aligned}
 &V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \cos \psi_1 \\
 &+ V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \cos \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \sin \psi_1 \\
 &= -\tan \phi \left[V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} + V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \cos \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \cos \psi_1 \right. \\
 &\quad \left. + V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin \left(\frac{\psi \omega}{\Omega} - \alpha_1 \right) \sin \psi_1 \right].
 \end{aligned}$$

For the ion to be accelerated without precession for varying ϕ the above equations must be independent of ϕ . Hence the coefficients of $\tan \phi$ above must be made zero. When the restriction on α_1 prescribed above in equation (11) is introduced into these coefficients, they become:

$$V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin \psi_1 = 0$$

and

(16)

$$V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} = -V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \cos \psi_1.$$

These equations are satisfied if:

$$(a) \psi_1 = \pi, \text{ and } \frac{V_1}{V_2} = \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega}}{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega}} \quad (17)$$

or

$$(b) \psi_1 \neq \pi, \text{ and } \theta_1 = \frac{(2c+1)\pi}{\omega/\Omega} \text{ and } \theta_2 = \frac{(2d+1)\pi}{\omega/\Omega}, \quad c, d = 1, 2, 3, \dots \quad (18)$$

With the right hand side of equations (15) thus made equal to zero the left hand side must be satisfied in order that ion does not precess from the cyclotron, according to condition (8). Again imposing the

restriction on α_1 in equation (11), these equations reduce to :

$$V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} + V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \cos \psi_1 = 0$$

and

(19)

$$V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin \psi_1 = 0.$$

These equations are satisfied if:

$$(a) \psi_1 = \pi, \text{ and } \frac{V_1}{V_2} = \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega}}{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega}} \quad (20)$$

or

(21)

$$(b) \psi_1 \neq \pi, \text{ and } \theta_1 = \frac{2c\pi}{\omega/\Omega} \text{ and } \theta_2 = \frac{2d\pi}{\omega/\Omega}, c, d, = 1, 2, 3, \dots$$

$$\text{Hence, clearly } \psi_1 \text{ must be } \pi \text{ and } \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega}}{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega}} = \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega}}{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega}} \quad (22)$$

since the solutions for $\psi_1 \neq \pi$ do not coincide.

2. Two non-similar dees, unevenly spaced.

It is possible to construct a cyclotron with $\psi_1 \neq \pi$. The phasing angle α_1 must be chosen differently and hence maximum energy gain per turn will not be attained. But in some cases, it might be advantageous to have $\psi_1 \neq \pi$ and sacrifice some energy gain per turn. Consider

$$\alpha_1 = (\frac{\omega}{\Omega} \neq 1) \psi_1 + (2s - 1)\pi \text{ where } s = 0, 1, 2, \dots \quad (23)$$

Then

$$\Delta W = 2q \left[V_1 \sin \frac{\theta_1 \omega}{2\Omega} \cos \phi - V_2 \sin \frac{\theta_2 \omega}{2\Omega} \cos (\phi - \psi_1) \right] \quad (24)$$

Again for $\phi = 0, \psi_1 = \pi$, maximum energy gain is attained which is identical with the previous case.

The coefficients of $\tan \phi$ in equation (15) become:

$$\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} = \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \quad (25)$$

and

$$\begin{aligned} V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} & \neq V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \cos^2 \psi_1 \\ & \neq V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin^2 \psi_1 = 0. \end{aligned} \quad (26)$$

Hence,

$$V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} = - V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \quad (27)$$

or

$$V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} = - V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega}.$$

Also equation (30) gives the requirement that:

$$\theta_2 = \frac{2d\pi}{\omega/\Omega - 1}, \quad \frac{\omega}{\Omega} \neq 1, \quad d = \text{integer}. \quad (28)$$

For

$$\frac{\omega}{\Omega} = 1, \text{ there is no restriction on } \theta_2.$$

For non-precession of the ion in its orbit, the following hold:

$$\begin{aligned} V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} & \neq V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \cos^2 \psi_1 \\ & \neq V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin^2 \psi_1 = 0 \end{aligned} \quad (29)$$

and

$$\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} = \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega}. \quad (30)$$

Hence,

$$V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} = - V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \quad (31)$$

or

$$V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} = - V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega}.$$

Again from equation (30):

$$\theta_2 = \frac{2d\pi}{\omega/\Omega - 1} \text{ for } \frac{\omega}{\Omega} \neq 1, d = \text{an integer.} \quad (32)$$

For $\frac{\omega}{\Omega} = 1$, there is no restriction on θ_2 .

So, in order to satisfy both sets of equations, the following must be true:

$$(a) \theta_1 = \frac{2c\pi}{\omega/\Omega - 1} \text{ and } \theta_2 = \frac{2d\pi}{\omega/\Omega - 1} \text{ for } c, d \text{ integers and } \frac{\omega}{\Omega} \neq 1. \quad (33)$$

For $\frac{\omega}{\Omega} = 1$, there is no restriction on θ_1 or θ_2 .

$$(b) \frac{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega}}{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega}} = \frac{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega}}{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega}} \quad (34)$$

The special case of $\theta_1 = \theta_2$ satisfies the above for all values of $\frac{\omega}{\Omega}$.

An interesting case of this phasing condition (23) is found for $\psi_1 \leq \frac{\pi}{2}$. For simplicity, consider $\psi_1 = \frac{\pi}{2}$ and $\theta_1 = \theta_2 < \frac{\pi}{2}$ in Figure 3. Then

$$\alpha_1 = \left(\frac{\omega}{\Omega} \neq 1\right) \psi_1 \neq (2s-1)\pi \neq 0$$

for $\frac{\omega}{\Omega} = 4c \neq 1, c = 0, 1, 2, \dots$, and $s = 0, \neq 1, \neq 2, \dots$

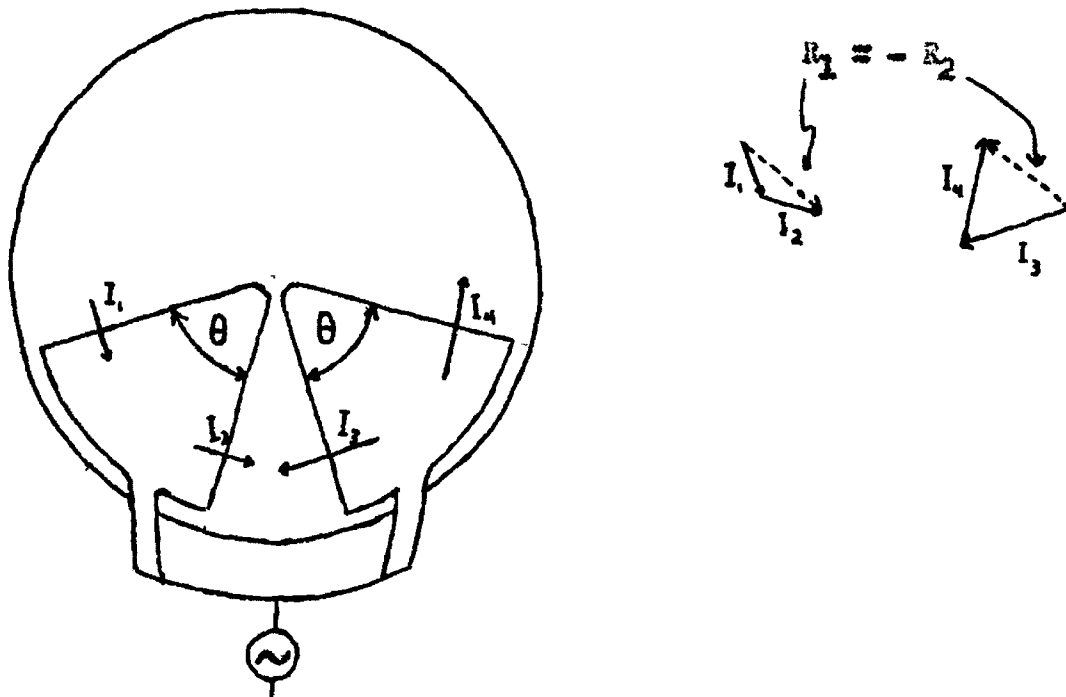


Figure 3.

These two dees occupy the same general area as does ³one-dee cyclotron with dee width ($\psi_1 \neq 0$). Yet a one-dee cyclotron of that width will not work, since the ion will precess from the machine. No precession occurs here, i.e. $-R_1 = R_2$, even though the dees are driven in phase. However, the energy gain per turn is only one half of the maximum attainable for $\psi_1 = \pi$.

In conclusion then, it is shown that for any ψ_1 , within the physical limits of the machine, acceleration can be obtained for ions of $\frac{\omega}{\Omega} = 1$ for any choice of θ_1 and θ_2 , provided the voltage phasing is chosen properly. The dee angles and dee separation must be correlated to prevent overlap of the dees. Maximum energy gain is obtained for $\psi_1 = \pi$. In general, for $\frac{\omega}{\Omega} > 1$ and variable spacing of the dees, it is necessary to construct the dees of certain specified widths for each $\frac{\omega}{\Omega}$. There are, of course, sets of harmonics for which the same dee widths can be utilized. In order to attain stable acceleration for all $\frac{\omega}{\Omega}$ without changing dee widths, it is necessary to impose the restriction that $\theta_1 = \theta_2$. If $\psi_1 = \pi$ and the above restriction is imposed, then the symmetrical two-dee case is obtained. The two voltage phasing relations (11) and (23) are now identical and by using them for all values of s , ions of almost all $\frac{\omega}{\Omega}$ ratios can be accelerated for a given value of $\theta_1 = \theta_2$. Those which cannot be accelerated are those which make the $\sin \frac{\theta \omega}{2\Omega}$ term zero in equation (12).

3. The three-dee case.

Consider the three-dee cyclotron sketched in Figure 4. It has the following characteristics:

$$\theta_1 \neq \theta_2 \neq \theta_3$$

$$\psi_1 \neq \psi_2 \neq \psi_3$$

$$V_1 \neq V_2 \neq V_3$$

$$\alpha_1 \neq \alpha_2 \neq \alpha_3$$

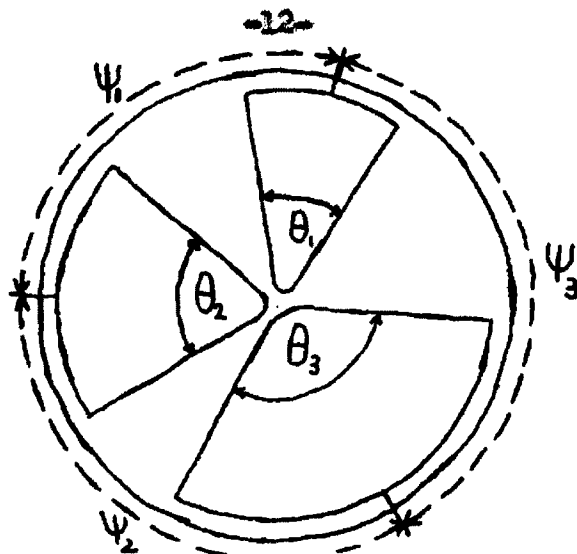


Figure 4.

By the same method as above, the energy gain per turn then is shown to be:

$$\Delta W = 2q \left[V_1 \sin \frac{\theta_1 \omega}{2\Omega} \cos \phi + V_2 \sin \frac{\theta_2 \omega}{2\Omega} \cos \left(\phi + \frac{\psi_1 \omega}{\Omega} - \alpha_1 \right) + V_3 \sin \frac{\theta_3 \omega}{2\Omega} \cos \left(\phi + (\psi_1 + \psi_2) \frac{\omega}{\Omega} - (\alpha_1 + \alpha_2) \right) \right] \quad (35)$$

By the same method as in the two-dee cyclotron, it can readily be shown that ΔW will be maximum only if the voltage phasing varies with the dee separation according to:

$$\begin{aligned} \alpha_1 &= \psi_1 \frac{\omega}{\Omega} + 2s\pi \\ \alpha_2 &= \psi_2 \frac{\omega}{\Omega} + 2s\pi \end{aligned} \quad (36)$$

and

$$\alpha_3 = \psi_3 \frac{\omega}{\Omega} + 2s\pi, \quad s = 0, \pm 1, \pm 2, \dots$$

The maximum energy gain per turn is obtained under these restrictions for $\phi = 0$:

$$\Delta W_m = 2q \left[V_1 \sin \frac{\theta_1 \omega}{2\Omega} + V_2 \sin \frac{\theta_2 \omega}{2\Omega} + V_3 \sin \frac{\theta_3 \omega}{2\Omega} \right]. \quad (37)$$

Now the condition (8) above must be taken into account. Again by the same method as in the two-dee case the x component equation and the y component equation are obtained. In general form with α_j independent of ψ_j , the equations are long and cumbersome, but since maximum energy

gain is desirable, they are simplified by the restrictions on α_j in equation (36). The simplified x component equation can be written as:

$$\begin{aligned} & V_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} + V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \cos \psi_1 \\ & + V_3 \cos \frac{\theta_3}{2} \sin \frac{\theta_3 \omega}{2\Omega} \cos (\psi_1 + \psi_2) = \tan \phi \left[V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin \psi_1 \right. \\ & \left. + V_3 \sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2\Omega} \sin (\psi_1 + \psi_2) \right]. \end{aligned} \quad (38)$$

The simplified y component equation can be written as:

$$\begin{aligned} & V_2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin \psi_1 + V_3 \cos \frac{\theta_3}{2} \sin \frac{\theta_3 \omega}{2\Omega} \sin (\psi_1 + \psi_2) \\ & = -\tan \phi \left[V_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} + V_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \cos \psi_1 \right. \\ & \left. + V_3 \sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2\Omega} \cos (\psi_1 + \psi_2) \right]. \end{aligned} \quad (39)$$

Again, for the ion to be accelerated without precession for varying ϕ , the above equation (38) and (39) must be independent of ϕ . The coefficients of the terms in $\tan \phi$ are made zero. When this is done, voltage ratios are obtained that must be satisfied:

$$\frac{V_1}{V_2} = \frac{-\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin \psi_2}{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} \sin (\psi_1 + \psi_2)}$$

and

$$\frac{V_1}{V_3} = \frac{\sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2\Omega} \sin \psi_2}{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} \sin \psi_1}.$$

(40)

Additional voltage ratios necessary to ensure that condition (3) above be satisfied are obtained by examining the left hand sides of

equations (38) and (39) when the right hand sides are made zero. Thus:

$$\frac{v_1}{v_2} = \frac{-\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2 \Omega} \sin \psi_2}{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2 \Omega} \sin (\psi_1 + \psi_2)}$$

and

(41)

$$\frac{v_1}{v_2} = \frac{\cos \frac{\theta_3}{2} \sin \frac{\theta_3 \omega}{2 \Omega} \sin \psi_2}{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2 \Omega} \sin \psi_1}.$$

Therefore to satisfy both requirements of non-precession the following relations must be met:

$$\frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2 \Omega}}{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2 \Omega}} = \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2 \Omega}}{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2 \Omega}}$$

and

(42)

$$\frac{\cos \frac{\theta_3}{2} \sin \frac{\theta_3 \omega}{2 \Omega}}{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2 \Omega}} = \frac{\sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2 \Omega}}{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2 \Omega}}.$$

In contrast to the two-dee cyclotron, ions can be accelerated to maximum energy gain per turn for any value of ψ_1 , ψ_2 , and ψ_3 except π . When one of the angles of separation is 180° , the condition for non-precession of the ion in its orbit cannot be met. The angular extent of the dees is not arbitrary, except when ions of ratio $\frac{\omega}{\Omega} = 1$ are being accelerated. All cases of stable acceleration for $\frac{\omega}{\Omega} > 1$ must satisfy equation (42) above. Clearly, the simplest case of θ_1

$\theta_2 = \theta_3$ meets this requirement. For unequal shelves of θ_j , the widths of the dees would have to be changed in order to accelerate ions of different $\frac{\omega}{\Omega}$ ratio. This, of course, would be a distinct limitation on this type of machine.

4. The four-dee case.

In an analogous manner as that of the two-dee case the energy gain per turn for a four-dee-cyclotron is found to be

$$\Delta W = 2q \cos \phi \left[V_1 \sin \frac{\theta_1 \omega}{2\Omega} + V_2 \sin \frac{\theta_2 \omega}{2\Omega} + V_3 \sin \frac{\theta_3 \omega}{2\Omega} + V_4 \sin \frac{\theta_4 \omega}{2\Omega} \right] \quad (43)$$

where again for maximum energy gain per turn α_j is restricted as follows:

$$\alpha_j = \frac{\omega}{\Omega} \psi_j \neq 2\pi \quad (44)$$

The energy gain equation and voltage ratios can be expressed in general without making the restriction, but they are very involved expressions. The summation of the x components and y components of the impulses given an ion in one revolution are again equal to zero, if there is to be no precession of the ion in its orbit. By the same argument as before, the voltage relations are obtained. In these relations that follow, note that two of the voltages are arbitrary, and then the other two are specified by these two:

$$V_3 = V_1 \frac{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} \sin \psi_4}{\cos \frac{\theta_2}{2} \sin \frac{\theta_3 \omega}{2\Omega} \sin \psi_3} + V_2 \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin (\psi_1 + \psi_4)}{\cos \frac{\theta_3}{2} \sin \frac{\theta_3 \omega}{2\Omega} \sin \psi_3} \quad (45)$$

$$V_4 = V_1 \frac{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} \sin (\psi_1 + \psi_2)}{\cos \frac{\theta_4}{2} \sin \frac{\theta_4 \omega}{2\Omega} \sin \psi_3} + V_2 \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin \psi_2}{\cos \frac{\theta_4}{2} \sin \frac{\theta_4 \omega}{2\Omega} \sin \psi_3}$$

Then, in order that there be no net impulse per turn for various phase angle ϕ the voltage relations must be:

$$V_3 = V_1 \frac{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} \sin \psi_4}{\sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2\Omega} \sin \psi_3} \neq V_2 \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin (\psi_1 \neq \psi_4)}{\sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2\Omega} \sin \psi_3}$$

and

$$V_4 = V_1 \frac{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} \sin (\psi_1 \neq \psi_2)}{\sin \frac{\theta_4}{2} \cos \frac{\theta_4 \omega}{2\Omega} \sin \psi_3} \neq V_2 \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2\Omega} \sin \psi_2}{\sin \frac{\theta_4}{2} \cos \frac{\theta_4 \omega}{2\Omega} \sin \psi_3} \quad (46)$$

The four-dee cyclotron must be of such geometry or the ion of such $\frac{\omega}{\Omega}$ ratio that the dependent voltages, V_3 and V_4 , satisfy both of the requirements above. The conditions under which these are met are identical with those listed previously in the summary for the three-dee case. In addition, for four-dees, stable acceleration can be maintained by varying the ψ_j 's, as well as θ_j 's, if the ratio $\frac{\omega}{\Omega}$ is changed to values > 1 .

5. The N-dee case.

As an extension of the equations given above for the four-dee case, the following hold for the N-dee case. Again, to simplify otherwise very long and cumbersome expressions and also since maximum energy gain per turn is desirable, α_j is restricted as in the four-dee case. The energy gain per turn is:

$$\Delta W \approx 2q \cos \phi \left[V_1 \sin \frac{\theta_1 \omega}{2\Omega} \neq V_2 \sin \frac{\theta_2 \omega}{2\Omega} \neq \dots \neq V_{N-1} \sin \frac{\theta_{N-1} \omega}{2\Omega} \right] \quad (47)$$

where V_1, \dots, V_{N-2} are arbitrarily chosen for a given set of ψ_j .

V_{N-1} and V_N are given by the following relations in order that condition (8) be met.

$$V_{N-1} = V_1 \frac{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} \sin \psi_N}{\cos \frac{\theta_{N-1}}{2} \sin \frac{\theta_{N-1} \omega}{2\Omega} \sin \psi_{N-1}} \neq V_2 \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin (\psi_1 \neq \psi_N)}{\cos \frac{\theta_{N-1}}{2} \sin \frac{\theta_{N-1} \omega}{2\Omega} \sin \psi_{N-1}}$$

$$\neq V_3 \frac{\cos \frac{\theta_3}{2} \sin \frac{\theta_3 \omega}{2\Omega} \sin (\psi_1 \neq \psi_2 \neq \psi_N)}{\cos \frac{\theta_{N-1}}{2} \sin \frac{\theta_{N-1} \omega}{2\Omega} \sin \psi_{N-1}} \neq \dots$$

$$\neq V_{N-2} \frac{\cos \frac{\theta_{N-2}}{2} \sin \frac{\theta_{N-2} \omega}{2\Omega} \sin (\psi_1 \neq \dots \neq \psi_{N-3} \neq \psi_N)}{\cos \frac{\theta_{N-1}}{2} \sin \frac{\theta_{N-1} \omega}{2\Omega} \sin \psi_{N-1}}$$

and

(45)

$$V_N = V_1 \frac{\cos \frac{\theta_1}{2} \sin \frac{\theta_1 \omega}{2\Omega} \sin (\psi_1 \neq \psi_2 \neq \dots \neq \psi_{N-2})}{\cos \frac{\theta_N}{2} \sin \frac{\theta_N \omega}{2\Omega} \sin \psi_{N-1}}$$

$$\neq V_2 \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_2 \omega}{2\Omega} \sin (\psi_1 \neq \dots \neq \psi_{N-2})}{\cos \frac{\theta_N}{2} \sin \frac{\theta_N \omega}{2\Omega} \sin \psi_{N-1}} \neq \dots$$

$$\neq V_{N-2} \frac{\cos \frac{\theta_{N-2}}{2} \sin \frac{\theta_{N-2} \omega}{2\Omega} \sin \psi_{N-2}}{\cos \frac{\theta_N}{2} \sin \frac{\theta_N \omega}{2\Omega} \sin \psi_{N-1}} .$$

In order that there be no net impulse per turn for various ϕ the following voltage relations must be met:

$$V_{N-1} = V_1 \frac{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2\Omega} \sin \psi_N}{\sin \frac{\theta_{N-1}}{2} \cos \frac{\theta_{N-1} \omega}{2\Omega} \sin \psi_{N-1}}$$

$$\begin{aligned}
 & \neq V_2 \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2 \Omega} \sin (\psi_1 \neq \psi_N)}{\sin \frac{\theta_{N-1}}{2} \cos \frac{\theta_{N-1} \omega}{2 \Omega} \sin \psi_{N-1}} \\
 & \neq V_3 \frac{\sin \frac{\theta_3}{2} \cos \frac{\theta_3 \omega}{2 \Omega} \sin (\psi_1 \neq \psi_2 \neq \psi_N) \neq \dots}{\sin \frac{\theta_{N-1}}{2} \cos \frac{\theta_{N-1} \omega}{2 \Omega} \sin \psi_{N-1}} \\
 & \neq V_{N-2} \frac{\sin \frac{\theta_{N-2}}{2} \cos \frac{\theta_{N-2} \omega}{2 \Omega} \sin (\psi_1 \neq \dots \neq \psi_{N-3} \neq \psi_N)}{\sin \frac{\theta_{N-1}}{2} \cos \frac{\theta_{N-1} \omega}{2 \Omega} \sin \psi_{N-1}}
 \end{aligned}$$

and

$$\begin{aligned}
 V_N &= V_1 \frac{\sin \frac{\theta_1}{2} \cos \frac{\theta_1 \omega}{2 \Omega} \sin (\psi_1 \neq \psi_2 \neq \dots \neq \psi_{N-2})}{\sin \frac{\theta_N}{2} \cos \frac{\theta_N \omega}{2 \Omega} \sin \psi_{N-1}} \quad (49) \\
 & \neq V_2 \frac{\sin \frac{\theta_2}{2} \cos \frac{\theta_2 \omega}{2 \Omega} \sin (\psi_2 \neq \dots \neq \psi_{N-2}) \neq \dots}{\sin \frac{\theta_N}{2} \cos \frac{\theta_N \omega}{2 \Omega} \sin \psi_{N-1}} \\
 & \neq V_{N-2} \frac{\sin \frac{\theta_{N-2}}{2} \cos \frac{\theta_{N-2} \omega}{2 \Omega} \sin \psi_{N-2}}{\sin \frac{\theta_N}{2} \cos \frac{\theta_N \omega}{2 \Omega} \sin \psi_{N-1}} .
 \end{aligned}$$

In order to obtain a dependent voltage satisfying both conditions, the right hand sides of the two V_{N-1} relations and the two V_1 relations must be equal.

In general the conclusions for the three-dee and the four-dee non-symmetrical cyclotron can be extended to the N-dee case, with one addition. For $N \geq 5$, ions of different values of $\frac{\omega}{\Omega} > 1$ can be accelerated without precession by proper adjustment of from one to (N-4) of the

arbitrary voltages. This means of compensating for change in $\frac{\omega}{\Omega}$ ratio is much easier to accomplish than the requirements for change of θ_j 's and ψ_j 's in the cases of $n = 3$ and 4.

CONCLUSIONS

In a symmetrical N-dee cyclotron the energy gain per turn can be increased considerably by introducing a larger number of dees with proper voltage phasing. The peak voltage phase shift is the same for each dee. The threshold voltage is defined as the minimum voltage necessary to bring an ion out to the exit radius. By derivation this threshold voltage can be shown to be inversely proportional to the energy gain per turn. Hence the voltage and power necessary to drive the dees can be reduced by increasing the energy gain per turn. Table I lists the energy gain for various ions and several values of n . By increasing the energy gain per turn the number of ions reaching the exit radius with a given energy is increased. This is because fewer turns will be required, and thus fewer ions will be lost by phase variation and by ions striking the dees. On the other hand, this increasing beam will probably be offset, to some extent, by the requirement for a source considerably off center. A wide variety of ions could be accelerated in a multidee cyclotron simply by voltage phase adjustments. From consideration of the upper limit on the energy gain per turn as n approaches infinity it is noted that ΔW is proportional to $\frac{\omega}{\Omega}$. Therefore the energy gain per turn increases as the $\frac{\omega}{\Omega}$ ratio increases. For example, for an ion of $\frac{\omega}{\Omega} = 1$, ΔW increases from $4qV$ at $n = 2$ to $6qV$ at $n = 6$ and $2\pi qV$ at $n = \infty$, but for an ion of $\frac{\omega}{\Omega} = 5$, ΔW increases

from $4qV$ at $n = 2$ to $6qV$ at $n = 6$ and $10\pi qV$ at $n = \infty$.

Other than the possibility of constructing a cyclotron to fit a certain prescribed geometry, the non-symmetrical machine presents no particular advantage over the symmetrical machine of equal number of dees. For equivalent voltages the energy gain per turn will never be greater than the gain for a symmetrical machine of maximum width dees. However, the maximum energy gain per turn is still desirable, although the threshold voltage must be redefined. Maximum energy gain per turn is obtained by proper choice of the voltage phasing between dees. Threshold voltage for a cyclotron with various peak voltages on different dees is defined for each dee as the voltage necessary on that dee to bring the ion out to the exit radius. These threshold voltages can be zero for all but three dees. These three voltages are chosen so that they satisfy the restrictions of non-precession as well as the definition of threshold voltage. The non-symmetrical machine has the serious limitation that only ions of one $\frac{\omega}{\Omega} > 1$ ratio can be accelerated in a particular machine. To accelerate ions of different $\frac{\omega}{\Omega} > 1$, it is necessary to adjust the angular extent of the dees for $N \geq 2$, the spacing of the dees for $N \geq 4$, or the voltages on the dees for $N \geq 5$. All types of multidee cyclotrons would probably give rise to increased constructional and operational problems. One such problem would be the difficulty in properly phasing the dee voltages at high frequencies.

TABLE I

Energy gain per turn for various symmetrical cyclotrons and various ions.

n = number of dees; θ = dee angle; $\frac{\omega}{\Omega}$ = rf angular frequency; Ω = ion frequency; and P = integer.

Number of dees n	Dee angle θ	Ions accelerated $\frac{\omega}{\Omega}$	Maximum energy gain per turn, ΔW_M
2	180°	$2P \neq 1$ $2P$	$4qV$ 0
3	120°	$3P \neq 1$ $3P \neq 2$ $3P$	$3\sqrt{3} qV$ $3\sqrt{3} qV$ 0
4	90°	$4P \neq 1$ $4P \neq 2$ $4P \neq 3$ $4P$	$4\sqrt{2} qV$ $8qV$ $4\sqrt{2} qV$ 0
6	60°	$6P \neq 1$ $6P \neq 2$ $6P \neq 3$ $6P \neq 4$ $6P \neq 5$ $6P$	$6qV$ $6\sqrt{3} qV$ $12qV$ $6\sqrt{3} qV$ $6qV$ 0
N	$\frac{2\pi}{N}$	$NP \neq 1$ $NP \neq 2$... NP	$2NqV \sin \left[\pi \left(P \neq \frac{1}{N} \right) \right]$ $2NqV \sin \left[\pi \left(P \neq \frac{2}{N} \right) \right]$... 0

DISCUSSION

Jakobson and Schmidt, "Characteristics of a Proposed Double-Mode Cyclotron", Physical Review, Volume 93, page 303 (1954).

Jakobson, Hussinkveld, and Ruby, "Modes of Acceleration of Ions in a Three-Dee Cyclotron", Physical Review, Volume 104, page 262 (1956).